The Law of Sines

**The Law of Sines**(or**Sine Rule**) is very useful for solving triangles:



It works for any triangle:

|  |  |
| --- | --- |
| triangle | **a**, **b** and **c** are sides.**A**, **B** and **C** are angles.*(Side a faces angle A, side b faces angle B and side c faces angle C).* |

So if you **divide side a by the sine of angle A** it is equal to **side b divided by the sine of angle B**, and also equal to **side c divided by the sine of angle C**

Sure ... ?

Well, let's do the calculations for a triangle I prepared earlier:

|  |  |
| --- | --- |
| 5,8,9 Triangle | **a/sin A** = 8 / sin (62.2°) = 8 / 0.885... = **9.04...****b/sin B** = 5 / sin (33.5°) = 5 / 0.552... = **9.06...****c/sin C** = 9 / sin (84.3°) = 9 / 0.995... = **9.05...** |

The answers are **almost the same!**
*(They would be****exactly****the same if I used perfect accuracy).*

So now you can see that:

a/sin A = b/sin B = c/sin C

How Do I Use It?

Let us see an example:

**Example: Calculate side "c"**



|  |  |  |
| --- | --- | --- |
| Law of Sines: |   | a/sin A = b/sin B = c/sin C |
|   |   |   |
| Put in the values we know: |   | a/sin A = 7/sin(35°) = c/sin(105°) |
|   |   |   |
| Ignore a/sin A (not useful to us): |   | 7/sin(35°) = c/sin(105°) |
|   |   |   |
| ***Now we use our algebra skills to rearrange and solve****:* |
|   |   |   |
| Swap sides: |   | c/sin(105°) = 7/sin(35°) |
|   |   |   |
| Multiply both sides by sin(105°): |   | c = ( 7 / sin(35°) ) × sin(105°) |
|   |   |   |
| Calculate: |   | c = ( 7 / 0.574... ) × 0.966... |
| Calculate: |   | c = **11.8** (to 1 decimal place) |

Finding an Unknown Angle

In the previous example we found an unknown side ...

... but we can also use the Law of Sines to find an **unknown angle**.

In this case it is best to turn the fractions upside down (**sin A/a** instead of **a/sin A**, etc):



**Example: Calculate angle B**



|  |  |  |
| --- | --- | --- |
| Start with: |   | sin A / a = sin B / b = sin C / c |
|   |   |   |
| Put in the values we know: |   | sin A / a = sin B / 4.7 = sin(63º) / 5.5 |
|   |   |   |
| Ignore "sin A / a": |   | sin B / 4.7 = sin(63º) / 5.5 |
|   |   |   |
| Multiply both sides by 4.7: |   | sin B = (sin63º/5.5) × 4.7 |
| Calculate: |   | sin B = 0.7614... |
|   |   |   |
| Inverse Sine: |   | B = sin-1(0.7614...) |
|   |   | B = **49.6º** |

Sometimes There Are Two Answers !

There is one **very** tricky thing you have to look out for:

Two possible answers.

|  |  |  |
| --- | --- | --- |
| Sine Law Ambiguous Case | Let us say you know angle **A**, and sides **a** and **b**.You could swing side **a** to left or right and come up with two possible results (a small triangle and a much wider triangle)Both answers are right! |   |

This only happens in the "[Two Sides and an Angle **not** between](http://www.mathsisfun.com/algebra/trig-solving-ssa-triangles.html)" case, and even then not always, but you have to watch out for it.

Just think "could I swing that side the other way to also make a correct answer?"

**Example: Calculate angle R**



The first thing to notice is that this triangle has different labels: PQR instead of ABC. But that's not a problem. We just use P,Q and R instead of A, B and C in The Law of Sines.

|  |  |  |
| --- | --- | --- |
| Start with: |   | sin R / r = sin Q / q |
|   |   |   |
| Put in the values we know: |   | sin R / 41 = sin(39º)/28 |
|   |   |   |
| Multiply both sides by 41: |   | sin R = (sin39º/28) × 41 |
| Calculate: |   | sin R = 0.9215... |
|   |   |   |
| Inverse Sine: |   | R = sin-1(0.9215...) |
|   |   | R = **67.1º** |

But wait! There's another angle that also has a sine equal to 0.9215...

**Your calculator won't tell you this** but sin(112.9º) is also equal to 0.9215... (try it!)

So ... how do you discover the vale 112.9º?

Easy ... take 67.1º away from 180°, like this:

180° - 67.1° = 112.9°

So there are two possible answers for R: **67.1º** and **112.9º**:



Both are possible! Each one has the 39º angle, and sides of 41 and 28.

So, always check to see whether the alternative answer makes sense.

* ... sometimes it will (like above) and there will be**two solutions**
* ... sometimes it won't (see below) and there is **one solution**

|  |  |
| --- | --- |
| triangle | We looked at this triangle before.As you can see, you can try swinging the "5.5" line around, but no other solution makes sense.So this has only one solution. |

The Law of Cosines

**The Law of Cosines** (also called the **Cosine Rule**) is very useful for solving triangles:



It works for any triangle:

|  |  |
| --- | --- |
| http://www.mathsisfun.com/algebra/images/trig-sine-rule.gif | **a**, **b** and **c** are sides.**C** is the angle opposite side c |

Let's see how to use it in an example:

**Example: How long is side "c" ... ?**

****

We know angle C = 37º, a = 8 and b = 11

|  |  |  |
| --- | --- | --- |
| **The Law of Cosines** says: |   | c2 = a2 + b2 - 2ab cos(C) |
|   |   |   |
| Put in the values we know: |   | c2 = 82 + 112 - 2 × 8 × 11 × cos(37º) |
|   |   |   |
| Do some calculations: |   | c2 = 64 + 121 - 176 × 0.798… |
|   |   |   |
| Which gives us: |   | c2 = 44.44... |
|   |   |   |
| Take the square root: |   | c = √44.44 = 6.67 (to 2 decimal places) |

Answer: c = 6.67

How to Remember

How can you remember the formula?

Well, it helps to know that it is the [Pythagoras Theorem](http://www.mathsisfun.com/pythagoras.html) with something extra so it works for all triangles:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Pythagoras Theorem: |   | a2 + b2 = c2 |   | (only for Right-Angled Triangles) |
|   |   |   |   |   |
| Law of Cosines: |   | a2 + b2 - 2ab cos(C) = c2 |   | (for all triangles) |

So, to remember it:

* think "abc": **a**2 + **b**2 = **c**2,
* then another "abc": 2**ab** cos(**C**),
* and put them together: **a2 + b2 - 2ab cos(C) = c2**

When to Use

The law of cosines is useful for finding:

* the third side of a triangle when you know **two sides and the angle between** them (like the example above)
* the angles of a triangle when you know **all three sides** (as in the following example)

**Example: What is Angle "C" ...?**



The side of length "8" is opposite angle ***C***, so it is side **c**. The other two sides are **a** and **b**.

Now let us put what we know into **The Law of Cosines**:

|  |  |  |
| --- | --- | --- |
| Start with: |   | c2 = a2 + b2 - 2ab cos(C) |
|   |   |   |
| Put in a, b and c |   | 82 = 92 + 52- 2 × 9 × 5 × cos(C) |
|   |   |   |
| Calculate: |   | 64 = 81 + 25- 90 × cos(C) |
| Calculate some more: |   | 64 = 106 - 90 × cos(C) |
|   |   |   |
| ***Now we use our algebra skills to rearrange and solve****:* |
|   |   |   |
| Subtract 64 from both sides: |   | 0 = 42 - 90 × cos(C) |
|   |   |   |
| Add "90 × cos(C)" to both sides: |   | 90 × cos(C) = 42 |
|   |   |   |
| Divide both sides by 90: |   | cos(C) = 42/90 |
|   |   |   |
| Inverse cosine: |   | C = cos-1(42/90) |
|   |   |   |
| Calculator: |   | C = **62.2°** (to 1 decimal place) |

In Other Forms

Easier Version For Angles

There is a version that is easier to use when finding angles. It is simply a rearrangement of the c2 = a2 + b2 - 2ab cos(C) formula like this:



**Example: Find Angle "C"**



In this triangle we know the three sides:

* a = 8,
* b = 6 and
* c = 7.

Use The Law of Cosines (angle version) to find angle **C** :

cos C = (a² + b² - c²)/2ab

= (8² + 6² - 7²)/2×8×6 = (64 + 36 - 49)/96 = 51/96 = 0.53125

C = cos-1(0.53125)

= **57.9°** correct to one decimal place.

Versions for a, b and c

Also, you can rewrite the c2 = a2 + b2 - 2ab cos(C) formula into "a2=" and "b2=" form.

Here are all three:







But it is easier to remember the "**c2**=" form and change the letters as needed !

As in this example:

**Example: Find the distance "z"**



The letters are different! But that doesn't matter. We can easily substitute x for a, y for b and z for c

|  |  |  |
| --- | --- | --- |
| Start with: |   | c2 = a2 + b2 - 2ab cos(C) |
|   |   |   |
| x for a, y for b and z for c |   | z2 = x2 + y2 - 2xy cos(Z) |
|   |   |   |
| Put in the values we know: |   | z2 = 9.42 + 6.52 - 2×9.4×6.5×cos(131º) |
| Calculate: |   | z2 = 88.36 + 42.25 - 122.2×(-0.656...) |
|   |   | z2 = 130.61 + 80.17... |
|   |   | z2 = 210.78... |
|   |   | z = √210.78... = 14.5 to 1 decimal place. |

Answer: z = 14.5

Did you notice that cos(131º) is negative and this changes the last sign in the calculation to +(plus)? The cosine of an obtuse angle is always negative (see [Unit Circle](http://www.mathsisfun.com/geometry/unit-circle.html)).